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Steady reflection, absorption and transmission of small disturbances by a screen of dusty gas

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An analytical solution of a system of linearized equations for a gas–particle mixture is obtained for steady periodic motions. A finite volume fraction of particles and a continuous distribution of particle sizes are taken into consideration. It is shown that the effect of a continuous distribution of particle radii on the acoustic motions of a dusty gas is incorporated through only four integral quantities containing the relaxation times of the particle velocity and temperature, integrated over all particle sizes. This solution is applied to the problem of acoustic reflection, absorption, and transmission by a screen of dusty gas.

1. Introduction

When a small disturbance is incident upon a region where a gas and an appreciable amount of dust are present, reflection, absorption and transmission of the disturbance by the dusty gas will occur. Cutting off or shielding of acoustic waves by screens of a gas which contain much dust or many water droplets is a very important problem of both theoretical and practical interest. Recently, for example, at the lift-off of the U.S. Space Shuttle, a very strong water shower was used to suppress the strength of the ground-reflected waves produced by the exhaust jets. It was shown that this idea is quite effective in protecting the very weak wall structure of the Orbitor from possible fracture due to the air disturbances.

Determining the coefficients of reflection, absorption and transmission of waves by a screen of dusty gas requires the complete solution of the flow and relaxation equations for the gas–particle mixture. The system of governing equations for a dusty gas is very similar to that for vibrationally or chemically relaxing gases. The effects of vibrational and chemical relaxations on the wave phenomena are well understood, particularly in the linear regime (Vincenti 1959; Clarke 1960). There are, however, a few important differences between the relaxation phenomena in a pure gas and those in a dusty gas.

(*a*) The velocities of the particles can be, in general, different from the gas velocity, leading to distinct momentum equations for the gas and the particles that are coupled with each other. This is not the case for relaxing dust-free gases.

(*b*) Each particle has a finite volume, which introduces many coupling terms into the conservation equations of mass, momentum, and energy for the gas and particles except one (the energy-conservation equation for a particle).

(*c*) In many practical cases, the size of the particles is not always uniform. It is often considered to be a reasonably good approximation to assume that the particles have a continuous size distribution, which then implies an infinitely large number of different relaxation times for the particles.

These differences make it impossible for us to apply the previous solutions for relaxing dust-free gases directly to flows of gas–particle mixtures.

In this paper, the governing equations for a dusty gas are derived, taking into account a finite volume fraction and a continuous size distribution of the particles. The system of linearized equations is then applied to a one-dimensional problem where a finite-width screen of a uniform gas–particle mixture is present in a pure ideal gas and a specified wave is continually incident upon the gas–particle screen, as shown in figure 1. An analytical solution is obtained to determine the coefficients of reflection, absorption and transmission of the wave by the screen of dusty gas. Sample calculations are carried out for the gas–particle mixture composed of air and solid alumina (Al_2O_3) particles.

2. Assumptions

The gas–particle mixture is taken to be in an equilibrium reference state without disturbances. The analysis will be based on the following assumptions:

(i) particles are permanent;

(ii) the viscous force acting on each particle obeys Stokes' law;

(iii) the heat-transfer rate to each particle is proportional to the temperature difference between the gas and the particle;

(iv) the ratio of the gas density to the material density of the particles is smaller than unity;

(v) the gas is inviscid except for its interaction with the particles;

(vi) the gas is a perfect gas with constant composition and constant specific heats;

(vii) the thermal and Brownian motions of the particles are negligible;

(viii) the particles do not interact with each other;

(ix) the particles are solid or liquid spheres with a constant material density;

(x) the particles have a constant specific heat and the internal temperature of the particles is uniform.

These assumptions have been used in many previous papers (Takano & Adachi 1975; Rudinger 1970; Marble 1963; Carrier 1958). Here, however, the assumption of a single uniform diameter of particle sizes is not made. The assumption (iv) is quite reasonable in practice, and, as will be seen later, the ratio is important in determining the form of the general solution to the system.

3. The particle distribution function

The distribution function $\phi(x, t, r_p)$ is defined such that the number of particles that have radii r_p lying in the range r_p to $r_p + dr_p$ is

$$n_p \phi(x, t, r_p)\, dr_p \tag{1}$$

per unit volume, where n_p is the number density of the particles, and x and t are the distance and the time respectively. With this definition

$$n_p = \int n_p \phi(x, t, r_p)\, dr_p, \tag{2}$$

where the integration is taken over all sizes of particles contained in the flow, and it follows that the distribution function ϕ satisfies

$$\int \phi(x, t, r_p)\, dr_p = 1. \tag{3}$$

Also, the particle concentration σ_p can be written as

$$\sigma_p = \int m_p n_p \phi\, dr_p. \tag{4}$$

Here m_p is the mass of a particle and is given by

$$m_p = \tfrac{4}{3}\pi r_p^3 \rho_p, \tag{5}$$

where ρ_p is the material density of the particles. The volume fraction ϵ of the particles is also obtained as

$$\epsilon = n_p \int \tfrac{4}{3}\pi r_p^3 \phi\, dr_p, \tag{6}$$

from which the particle concentration σ_p given by (4) can be rewritten in the form

$$\sigma_p = \epsilon \rho_p. \tag{7}$$

By using the volume ratio ϵ, the gas concentration σ is given by

$$\sigma = (1-\epsilon)\rho, \tag{8}$$

where ρ is the gas density. An average particle radius l_p is defined for later convenience by

$$\tfrac{4}{3}\pi l_p^3 n_{p0} = \int \tfrac{4}{3}\pi r_p^3 n_{p0} \phi_0\, dr_p,$$

or

$$l_p^3 = \int r_p^3 \phi_0\, dr_p, \tag{9}$$

where the subscript zero denotes the equilibrium reference state.

4. The governing equations

The governing equations for the one-dimensional flow of a gas–particle mixture are written using the distribution function ϕ as follows:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x}(\sigma u) = 0, \tag{10}$$

$$\sigma \frac{Du}{Dt} + \frac{\partial p}{\partial x} + \tfrac{4}{3}\pi \rho_p n_p \int \frac{D_p u_p}{Dt} \phi r_p^3\, dr_p = 0, \tag{11}$$

$$\sigma \frac{D}{Dt}(\tfrac{1}{2}u^2 + h) + \tfrac{4}{3}\pi \rho_p n_p \int \frac{D_p}{Dt}(\tfrac{1}{2}u_p^2 + h_p)\phi r_p^3\, dr_p - \frac{\partial p}{\partial t} = 0, \tag{12}$$

$$p = R\rho T, \tag{13}$$

$$\frac{\partial}{\partial t}(n_p \phi) + \frac{\partial}{\partial x}(n_p \phi u_p) = 0, \tag{14}$$

$$\frac{D_p u_p}{Dt} = A_p(u - u_p) - \frac{1}{\rho_p}\frac{\partial p}{\partial x}, \tag{15}$$

$$\frac{\mathrm{D_p} T_\mathrm{p}}{\mathrm{D}t} = B_\mathrm{p}(T - T_\mathrm{p}), \tag{16}$$

where

$$h = \frac{\gamma}{\gamma - 1} RT = C_{pg} T, \tag{17}$$

$$h_\mathrm{p} = C_{p\mathrm{p}} T_\mathrm{p} + \frac{p}{\rho_\mathrm{p}}, \tag{18}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x}, \tag{19}$$

$$\frac{\mathrm{D_p}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_\mathrm{p} \frac{\partial}{\partial x}. \tag{20}$$

In these equations, u, p, T and h are the velocity, the pressure, the temperature and the specific enthalpy of the gas respectively. The roman subscript p denotes the quantities associated with the particles. The quantities R, C_{pg}, γ and $C_{p\mathrm{p}}$ are the gas constant, the specific heat at constant pressure, the ratio of specific heats of the gas, and the specific heat of the particle material.

In (14) and (15), A_p and B_p are given by

$$A_\mathrm{p} = \frac{9}{2} \frac{\mu}{\rho_\mathrm{p} r_\mathrm{p}^2} f_\mathrm{p}, \tag{21}$$

$$B_\mathrm{p} = 3 \frac{\mu}{P_\mathrm{r} \rho_\mathrm{p} r_\mathrm{p}^2} \frac{C_{pg}}{C_{p\mathrm{p}}} g_\mathrm{p}, \tag{22}$$

where μ is the coefficient of viscosity of the gas and P_r is the Prandtl number. The quantities f_p and g_p are defined by

$$f_\mathrm{p} = \frac{C_\mathrm{D}}{C_{\mathrm{D\ Stokes}}}, \tag{23}$$

$$g_\mathrm{p} = \frac{N_\mathrm{u}}{N_{\mathrm{u\ Stokes}}}, \tag{24}$$

where C_D is the drag coefficient and N_u is Nusselt number, and the subscript Stokes denotes the values evaluated in Stokes' theory, which are

$$C_{\mathrm{D\ Stokes}} = \frac{24}{R_\mathrm{e}}, \tag{25}$$

$$N_{\mathrm{u\ Stokes}} = 2, \tag{26}$$

$$R_\mathrm{e} = \frac{2 r_\mathrm{p} \rho |u - u_\mathrm{p}|}{\mu}. \tag{27}$$

It is well known that, for very small Reynolds number $R_\mathrm{e} \ll 1$,

$$f_\mathrm{p} \approx 1, \tag{28}$$

$$g_\mathrm{p} \approx 1. \tag{29}$$

For acoustic propagation in air, the induced motions of the gas and also the particles are very small, which indicates that the Reynolds number R_e given by (27) may be very small for small sizes of particles. Therefore, in this paper it is assumed that the approximations (28) and (29) are valid (assumption (iv) in §2). In this case

$$B_p = \frac{2}{3P_r}\frac{C_{pg}}{C_{pp}}A_p. \tag{30}$$

5. The system of linearized equations

Since the flow is treated as a perturbation from an equilibrium reference flow, it is convenient to introduce non-dimensional quantities as follows.

$$\left.\begin{aligned} \frac{t}{\tau} = t', \quad \frac{x}{\tau a_{f0}} = x', \\ \frac{r_p}{l_p} = r'_p, \quad l_p\phi = \phi', \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} \frac{p}{p_0} = 1+p', \quad \frac{\rho}{\rho_0} = 1+\rho', \\ \frac{T}{T_0} = 1+T', \quad \frac{u}{a_{f0}} = u', \\ \frac{n_p}{n_{p0}} = 1+n'_p, \quad \frac{\phi}{\phi_0} = 1+\varphi', \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} \frac{\epsilon}{\epsilon_0} = 1+\eta', \quad \frac{T_p}{T_0} = 1+T'_p, \quad \frac{u_p}{a_{f0}} = u'_p, \\ \tau A_{p0} = A'_p, \quad \tau B_{p0} = B'_p, \end{aligned}\right\} \tag{33}$$

$$\nu = \frac{\sigma_{p0}}{\sigma_0} = \frac{\epsilon_0\rho_p}{(1-\epsilon_0)\rho_0}, \quad \theta = \frac{C_{pp}}{C_{pg}}, \tag{34}$$

where τ is a characteristic time to be specified later and a_{f0} is the speed of sound of the frozen gas–particle mixture. It is well known that

$$a_f^2 = \gamma\frac{p}{\rho}. \tag{35}$$

Here it is important to notice that, under the assumptions (28) and (29), the non-dimensional quantities A'_p and B'_p are functions of r'_p only to the first approximation, because the Prandtl number P_r is usually well approximated as a constant, and the coefficient of gas viscosity μ is, in many practical cases, a function only of the gas temperature.

With these non-dimensional quantities, the basic equations (3), (6) and (9)–(16) are rearranged in conjunction with (8) and (17)–(20) to yield to the first approximation

$$\int \phi'_0 \mathrm{d}r'_p = 1, \tag{36}$$

$$\epsilon_0 = \tfrac{4}{3}\pi n_{p0} l_p^3, \tag{37}$$

$$\int r_p'^3 \phi'_0 \mathrm{d}r'_p = 1, \tag{38}$$

$$\int \varphi' \phi_0' \mathrm{d}r_\mathrm{p}' = 0, \tag{39}$$

$$\eta' = n_\mathrm{p}' + \int \varphi' r_\mathrm{p}'^3 \phi_0' \mathrm{d}r_\mathrm{p}', \tag{40}$$

$$\frac{\partial \rho'}{\partial t'} - \frac{\epsilon_0}{1-\epsilon_0}\frac{\partial \eta'}{\partial t'} + \frac{\partial u'}{\partial x'} = 0, \tag{41}$$

$$\frac{\partial u'}{\partial t'} + \frac{1}{\gamma(1-\epsilon_0)}\frac{\partial p'}{\partial x'} + \nu \int \frac{\partial u_\mathrm{p}'}{\partial t'} \phi_0' r_\mathrm{p}'^3 \mathrm{d}r_\mathrm{p}' = 0, \tag{42}$$

$$\frac{\partial p'}{\partial t'} - \gamma \frac{\partial \rho'}{\partial t'} + \gamma\nu\theta \int \frac{\partial T_\mathrm{p}'}{\partial t'} r_\mathrm{p}'^3 \phi_0' \mathrm{d}r_\mathrm{p}' = 0, \tag{43}$$

$$p' = \rho' + T', \tag{44}$$

$$\frac{\partial n_\mathrm{p}'}{\partial t'} + \frac{\partial \varphi'}{\partial t'} + \frac{\partial u_\mathrm{p}'}{\partial x'} = 0, \tag{45}$$

$$\frac{\partial u_\mathrm{p}'}{\partial t'} + A_\mathrm{p}' u_\mathrm{p}' = A_\mathrm{p}' u' - \frac{\epsilon_0}{1-\epsilon_0}\frac{1}{\gamma\nu}\frac{\partial p'}{\partial x'}, \tag{46}$$

$$\frac{\partial T_\mathrm{p}'}{\partial t'} + B_\mathrm{p}' T_\mathrm{p}' = B_\mathrm{p}' T'. \tag{47}$$

We have eight equations (40)–(47) for nine variables φ', η', ρ', u', p', T', n_p', u_p' and T_p'. Since n_p' and ϕ' appear only in the form $n_\mathrm{p}'+\varphi'$, this system is sufficient to solve for the seven variables and one pair $n_\mathrm{p}'+\varphi'$. It must be noticed that (39) is the normalizing condition for ϕ' and is used in order to obtain the solutions φ' and n_p'. When, for example, the system is solved to yield $\psi'(x', t', r_\mathrm{p}')$ as a solution for $n_\mathrm{p}'+\varphi'$,

$$n_\mathrm{p}' + \varphi' = \psi', \tag{48}$$

n_p' is first solved by

$$\int (n_\mathrm{p}'+\varphi')\phi_0' \mathrm{d}r_\mathrm{p}' = n_\mathrm{p}' = \int \psi'(x', t', r_\mathrm{p}')\,\mathrm{d}r_\mathrm{p}'.$$

Then it is given that

$$\varphi' = \psi' - \int \psi' \phi_0' \mathrm{d}r_\mathrm{p}'. \tag{49}$$

For the present purpose, however, it is not necessary to obtain the separate solutions φ' and n_p'.

6. The flow model and boundary conditions

One-dimensional flow is considered. The flow region is divided into three parts, $x \leqslant 0$, $0 \leqslant x \leqslant L$, and $L \leqslant x$ as in figure 1. In the regions $-\infty \leqslant x \leqslant 0$ and $L \leqslant x \leqslant \infty$, the same dust-free gas is present in a uniform state. In the region $0 \leqslant x \leqslant L$, a uniform gas–particle mixture is present in an equilibrium state without disturbances. The gas density and temperature are the same in all regions in the reference state.

In the present analysis, we prescribe a wave incident from negative infinity ($x = -\infty$) described by

$$u_\mathrm{i}' = C_\mathrm{i} \sin(\omega' t' - \beta_\mathrm{i} x'), \tag{50}$$

where ω' is a non-dimensional frequency, β_i is a constant, C is a non-dimensional

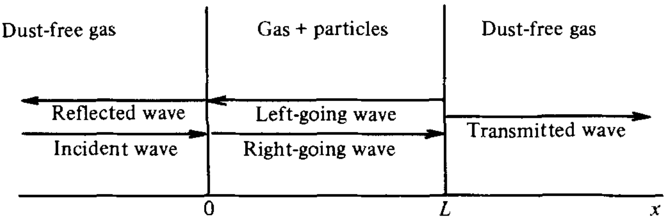

FIGURE 1. Flow model.

amplitude, and the subscript i denotes the incident wave. The non-dimensional frequency ω' is defined by

$$\omega' = \tau\omega = 1, \tag{51}$$

where ω is a dimensional frequency, and here the characteristic time τ introduced previously is defined as

$$\tau = \frac{1}{\omega}. \tag{52}$$

From the acoustic theory for a classical ideal gas it is given that

$$\beta_i = \omega' = 1. \tag{53}$$

For steady-state reflection, absorption and transmission, it is sufficient to consider four induced waves in the flow field: a reflected wave in the region $-\infty \leqslant x' \leqslant 0$, right- and left-going waves in the region $0 \leqslant x' \leqslant L'$, and a transmitted wave in the region $L' \leqslant x'$, where L' is the non-dimensional width of the gas–particle screen, defined by

$$L' = \frac{L}{a_{f0}\tau} = \frac{\omega L}{a_{f0}}. \tag{54}$$

In general, the reflection and transmission are accompanied by phase changes. Denoting them by δ_r and δ_t, the reflected and transmitted waves can be written

$$u'_r = C_r \sin(t' + \beta_r x' + \delta_r), \tag{55}$$

$$u'_t = C_t \sin(t' - \beta_t x' + \delta_t). \tag{56}$$

Obviously, in the present case we can put

$$\beta_r = \beta_t = \beta_i = 1. \tag{57}$$

The problem is now reduced to determining the constants C_r, C_t, δ_r and δ_t for the specified incident wave (50). In order to do this, it is necessary to solve the system of flow equations in the region $0 \leqslant x' \leqslant L'$ (or $0 \leqslant x \leqslant L$), together with the boundary conditions

$$\left.\begin{aligned} u'(t', -0) &= u'(t', +0), \\ p'(t', -0) &= p'(t', +0) \end{aligned}\right\} \quad (x' = 0), \tag{58}$$

$$\left.\begin{aligned} u'(t', L'-0) &= u'(t', L+0), \\ p'(t', L'-0) &= p'(t', L'+0) \end{aligned}\right\} \quad (x' = L'). \tag{59}$$

The solutions in the regions $-\infty \leqslant x' \leqslant 0$ and $L' \leqslant x' \leqslant \infty$ are well known in the classical theory, and are given by

$$\left.\begin{aligned} u' &= C_i \sin(t' - x') + C_r \sin(t' + x' + \delta_r), \\ p' &= \gamma[C_i \sin(t' - x') - C_r \sin(t' + x' + \delta_r)] \end{aligned}\right\} \quad (-\infty \leqslant x' \leqslant 0), \tag{60}$$

$$\left.\begin{aligned} u' &= C_t \sin(t'-x'+\delta_t), \\ p' &= \gamma C_t \sin(t'-x'+\delta_t) \end{aligned}\right\} \quad (L' \leqslant x' \leqslant \infty), \tag{61}$$

and they will be used in conjunction with (58) and (59) to obtain the acoustic solutions for u' and p' in the gas–particle mixture.

7. The solution in the gas–particle mixture

Combining (42) and (43) yields

$$\left(\frac{\partial}{\partial t}+A_p\right)u_p-\epsilon_0\int\frac{\partial u_p}{\partial t}\phi_0 r_p^3\,\mathrm{d}r_p=\left(\frac{\epsilon_0}{\nu}\frac{\partial}{\partial t}+A_p\right)u. \tag{62}$$

The primes will be omitted for simplicity except when this would lead to confusion. In conjunction with (40)–(42) and (45), we can get, from (44) and (47),

$$\left(\frac{\partial}{\partial t}+B_p\right)\frac{\partial^2 T_p}{\partial t\,\partial x}=-B_p\left[\gamma(1-\epsilon_0)\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right] \\ -B_p\int\left[\gamma\nu(1-\epsilon_0)\frac{\partial^2 u_p}{\partial t^2}-\frac{\epsilon_0}{1-\epsilon_0}\frac{\partial^2 u_p}{\partial x^2}\right]\phi_0 r_p^3\,\mathrm{d}r_p, \tag{63}$$

and also, from (43),

$$\left[(1-\epsilon_0)\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}\right]+\int\left[\nu(1-\epsilon_0)\frac{\partial^2 u_p}{\partial t^2}-\frac{\epsilon_0}{1-\epsilon_0}\frac{\partial^2 u_p}{\partial x^2}\right]\phi_0 r_p^3\,\mathrm{d}r_p \\ -\nu\theta\int\frac{\partial^2 T_p}{\partial t\,\partial x}\phi_0 r_p^3\,\mathrm{d}r_p=0. \tag{64}$$

Equations (62)–(64) constitute a set of equations for u, u_p and T_p. Although these have integral terms with respect to r_p, they are not integro-differential equations in the usual sense, because they do not contain any term with partial differentiation with respect to r_p. This system, however, cannot be reduced to a system of pure differential equations.

For the present purpose, it is sufficient to consider the solutions in the form

$$u=\mathrm{Re}\,(C(x)\,\mathrm{e}^{\mathrm{i}t}), \tag{65}$$

$$u_p=\mathrm{Re}\,(C_p(x,r_p)\,\mathrm{e}^{\mathrm{i}t}), \tag{66}$$

$$T_p=\mathrm{Re}\,(F_p(x,r_p)\,\mathrm{e}^{\mathrm{i}t}), \tag{67}$$

where C, C_p and F_p are the functions of x and r_p (or x alone) to be determined, and Re( ) designates the real part of ( ). For later convenience, some integral quantities are introduced by

$$\hat{C}_p(x)=\int C_p(x,r_p)\,\phi_0 r_p^3\,\mathrm{d}r_p, \tag{68}$$

$$\left.\begin{aligned} \hat{A}_{pk} &= \int\frac{A_p^k}{A_p^2+1}\phi_0 r_p^3\,\mathrm{d}r_p, \\ \hat{B}_{pk} &= \int\frac{B_p^k}{B_p^2+1}\phi_0 r_p^3\,\mathrm{d}r_p \end{aligned}\right\} \quad (k=0,1,2). \tag{69}$$

It is easy to see that

$$\hat{A}_{p2}=1-\hat{A}_{p0},\quad \hat{B}_{p2}=1-\hat{B}_{p0}. \tag{70}$$

The solution for u is obtained first. Substituting (65) and (66) into (62) yields after some manipulation

$$C_{\mathrm{p}} = \frac{1}{A_{\mathrm{p}}^2+1}\left\{\left(A_{\mathrm{p}}^2+\frac{\epsilon_0}{\nu}\right)C+\epsilon_0\hat{C}_{\mathrm{p}}+\mathrm{i}\left[\epsilon_0 A_{\mathrm{p}}\hat{C}_{\mathrm{p}}-\left(1-\frac{\epsilon_0}{\nu}\right)A_{\mathrm{p}}C\right]\right\}. \tag{71}$$

In this equation, $\hat{C}_{\mathrm{p}}$ is not yet determined. Equation (71) is substituted into (68), yielding

$$\hat{C}_{\mathrm{p}} = (\hat{C}_{\mathrm{p}1}+\mathrm{i}\hat{C}_{\mathrm{p}2})C, \tag{72}$$

where $\hat{C}_{\mathrm{p}1}$ and $\hat{C}_{\mathrm{p}2}$ are now functions of $\hat{A}_{\mathrm{p}k}$ ($k = 0, 1, 2$). Equation (71) with (72) represents a complete relation between C and C_{p}. Next, (65)–(67) are substituted for u, u_{p} and T_{p} into (63) to yield

$$F_{\mathrm{p}x} = -\frac{B_{\mathrm{p}}+\mathrm{i}B_{\mathrm{p}}^2}{B_{\mathrm{p}}^2+1}\left\{[\gamma(1-\epsilon_0)C+C_{xx}]+\left[\gamma\nu(1-\epsilon_0)\hat{C}_{\mathrm{p}}+\frac{\epsilon_0}{1-\epsilon_0}\hat{C}_{\mathrm{p}xx}\right]\right\}, \tag{73}$$

where

$$(\ )_x = \frac{\partial}{\partial x}(\ ), \quad (\ )_{xx} = \frac{\partial^2}{\partial x^2}(\ ). \tag{74}$$

Finally, by using (65)–(67) in conjunction with (71)–(73), we can get an ordinary differential equation for $C(x)$ with respect to x in the form

$$(a_1-\mathrm{i}a_2)C_{xx}+(a_3-\mathrm{i}a_4)C = 0, \tag{75}$$

where

$$\left.\begin{aligned} a_1 &= 1+\nu\theta\hat{B}_{\mathrm{p}2}+\frac{\epsilon_0}{1-\epsilon_0}[(1+\nu\theta\hat{B}_{\mathrm{p}2})\hat{C}_{\mathrm{p}1}+\nu\theta\hat{B}_{\mathrm{p}1}\hat{C}_{\mathrm{p}2}],\\ a_2 &= \nu\theta\hat{B}_{\mathrm{p}1}+\frac{\epsilon_0}{1-\epsilon_0}[\nu\theta\hat{B}_{\mathrm{p}1}\hat{C}_{\mathrm{p}1}-(1+\nu\theta\hat{B}_{\mathrm{p}2})\hat{C}_{\mathrm{p}2}],\\ a_3 &= (1-\epsilon_0)\{(1+\gamma\nu\theta\hat{B}_{\mathrm{p}2})+\nu[(1+\gamma\nu\theta\hat{B}_{\mathrm{p}2})\hat{C}_{\mathrm{p}1}+\gamma\nu\theta\hat{B}_{\mathrm{p}1}\hat{C}_{\mathrm{p}2}]\},\\ a_4 &= (1-\epsilon_0)\{\gamma\nu\theta\hat{B}_{\mathrm{p}1}+\nu[\gamma\nu\theta\hat{B}_{\mathrm{p}1}\hat{C}_{\mathrm{p}1}-(1+\gamma\nu\theta\hat{B}_{\mathrm{p}2})\hat{C}_{\mathrm{p}2}]\}.\end{aligned}\right\} \tag{76}$$

If we put

$$C = C_\lambda \mathrm{e}^{\lambda x}, \tag{77}$$

where C_λ is a constant, it follows from (75) that

$$\lambda^2 = P+\mathrm{i}Q, \tag{78}$$

where

$$P = -\frac{a_1a_3+a_2a_4}{a_1^2+a_2^2}, \quad Q = \frac{a_1a_4-a_2a_3}{a_1^2+a_2^2}. \tag{79}$$

Obviously P and Q are real constants. By introducing real constants α, β by

$$\lambda = \alpha+\mathrm{i}\beta, \tag{80}$$

the relations

$$\alpha^2-\beta^2 = P, \quad 2\alpha\beta = Q \tag{81}$$

are obtained. Since it can be shown that Q is positive (see appendix) so that α, β have same sign, we get

$$\lambda = \pm(\alpha+\mathrm{i}\beta), \tag{82}$$

where α and β are defined as positive quantities and are given by

$$(\alpha,\beta) = ([\tfrac{1}{2}(P+(P^2+Q^2)^{\frac{1}{2}})]^{\frac{1}{2}}, [\tfrac{1}{2}(-P+(P^2+Q^2)^{\frac{1}{2}})]^{\frac{1}{2}}). \tag{83}$$

Redefining λ as $\lambda = \alpha + i\beta$, previous discussions suggest a solution in the form

$$u = C_\lambda e^{it+\lambda x} + C_{-\lambda} e^{it-\lambda x}, \tag{84}$$

where C_λ and $C_{-\lambda}$ are complex. Since $C(x)e^{it}$ is a solution for the system, then $C^*(x)e^{-it}$ is also a solution for the system, where $C^*(x)$ is the complex conjugate of $C(x)$. The general solution to the present system can then be written as

$$u = [C_+ \sin(t-\beta x) + D_+ \cos(t-\beta x)]\,e^{-\alpha x} + [C_- \sin(t+\beta x) + D_- \cos(t+\beta x)]\,e^{+\alpha x}, \tag{85}$$

where C_+, C_-, D_+, and D_- are now real constants and the subscripts plus and minus denote waves propagating in the positive and negative directions respectively. These constants are to be determined from the boundary conditions (58) and (59). The solution (85) indicates that the waves in the mixture will decrease in amplitude with propagation, since the parameter α is positive.

Since the boundary conditions are given in terms of the gas velocity u and the pressure p, the solution for p is needed. Substituting (85) for u into (62)–(64), we can get, after some manipulation,

$$\begin{aligned} p = \frac{\gamma(1-\epsilon_0)}{\alpha^2+\beta^2}\{&[(KC_+ - MD_+)\sin(t-\beta x) + (MC_+ + KD_+)\cos(t-\beta x)]\,e^{-\alpha x} \\ &- [(KC_- - MD_-)\sin(t+\beta x) + (MC_- + KD_-)\cos(t+\beta x)]\,e^{+\alpha x}\}, \end{aligned} \tag{86}$$

where

$$\left.\begin{aligned} K &= (1+\nu\hat{C}_{p1})\beta - \nu\hat{C}_{p2}\alpha, \\ M &= (1+\nu\hat{C}_{p1})\alpha + \nu\hat{C}_{p2}\beta. \end{aligned}\right\} \tag{87}$$

8. Determination of the constants C_r, C_t, δ_r and δ_t

When the boundary conditions (58) and (59) are applied to the solutions (60), (61), (85) and (86), a total of eight algebraic equations are obtained for the eight unknowns C_r, C_t, δ_r, δ_t, C_+, C_-, D_+ and D_-. These are written as follows:

$$C_+ + C_- = C_i + C_r \cos\delta_r, \tag{88}$$

$$D_+ + D_- = C_r \sin\delta_r, \tag{89}$$

$$\frac{1-\epsilon_0}{\alpha^2+\beta^2}[K(C_+ - C_-) - M(D_+ - D_-)] = C_i - C_r\cos\delta_r, \tag{90}$$

$$\frac{1-\epsilon_0}{\alpha^2+\beta^2}[M(C_+ - C_-) + K(D_+ - D_-)] = -C_r\sin\delta_r, \tag{91}$$

$$(C_+\cos\beta L + D_+\sin\beta L)\,e^{-\alpha L} + (C_-\cos\beta L - D_-\sin\beta L)\,e^{+\alpha L} = C_t\cos(\delta_t - L), \tag{92}$$

$$(-C_+\sin\beta L + D_+\cos\beta L)\,e^{-\alpha L} + (C_-\sin\beta L + D_-\cos\beta L)\,e^{+\alpha L} = C_t\sin(\delta_t - L), \tag{93}$$

$$\begin{aligned} \frac{1-\epsilon_0}{\alpha^2+\beta^2}\{&[(KC_+ - MD_+)\cos\beta L + (MC_+ + KD_+)\sin\beta L]\,e^{-\alpha L} \\ &- [(KC_- - MD_-)\cos\beta L - (MC_- + KD_-)\sin\beta L)]\,e^{+\alpha L}\} = C_t\cos(\delta_t - L), \end{aligned} \tag{94}$$

$$\begin{aligned} \frac{1-\epsilon_0}{\alpha^2+\beta^2}\{&[(MC_+ + KD_+)\cos\beta L - (KC_+ - MD_+)\sin\beta L]\,e^{-\alpha L} \\ &- [(KC_- - MD_-)\sin\beta L + (MC_- + KD_-)\cos\beta L]\,e^{+\alpha L}\} = C_t\sin(\delta_t - L). \end{aligned} \tag{95}$$

Although it is quite easy to solve these equations, the results are very lengthy and will not be given here.

With the results for C_+, C_-, D_+ and D_-, the solutions for u_p and T_p in the gas–particle mixture are easily constructed. In the present analysis, however, we are mainly concerned with the reflection, absorption and transmission, so that the explicit results for u_p and T_p are not necessary.

The coefficients of reflection, transmission and absorption of the gas–particle screen are given by

$$\alpha_r = \frac{C_r^2}{C_i^2}, \tag{96}$$

$$\alpha_t = \frac{C_t^2}{C_i^2}, \tag{97}$$

$$\alpha_a = 1 - \frac{C_r^2 + C_t^2}{C_i^2} \tag{98}$$

respectively, where the subscript a denotes absorption. The last equation (98) was obtained from the conservation law for the acoustic energy. These three coefficients can be determined by using the solutions for (88)–(95) for the specified conditions of ρ_0, T_0, ϕ_0, n_{p0} and L (in the dimensional notation).

9. Stability of the gas–particle screen

The stability of the gas–particle screen against the acoustic disturbance is investigated. Eliminating C_r, C_t, δ_r and δ_t from (88)–(95), a system of equations is obtained:

$$\mathbf{\Lambda} \begin{pmatrix} C_+ \\ C_- \\ D_+ \\ D_- \end{pmatrix} = \begin{pmatrix} 0 \\ 2\kappa C_i \\ 0 \\ 0 \end{pmatrix}, \tag{99}$$

where

$$\kappa = \frac{\alpha^2+\beta^2}{1-\epsilon_0}\frac{1}{K}, \tag{100}$$

and $\mathbf{\Lambda}$ is a matrix with elements Λ_{ij} $(i, j = 1, 2, 3, 4)$,

$$\left.\begin{aligned}
&\Lambda_{11} = \frac{M}{K}, \quad \Lambda_{12} = -\Lambda_{11}, \quad \Lambda_{13} = \kappa+1, \quad \Lambda_{14} = \kappa-1, \\
&\Lambda_{21} = \Lambda_{13}, \quad \Lambda_{22} = \Lambda_{14}, \quad \Lambda_{23} = -\Lambda_{11}, \quad \Lambda_{24} = \Lambda_{11}, \\
&\Lambda_{31} = \left[(\kappa-1)\cos\beta L - \frac{M}{K}\sin\beta L\right]e^{-2\alpha L}, \\
&\Lambda_{32} = \left[(\kappa+1)\cos\beta L - \frac{M}{K}\sin\beta L\right], \\
&\Lambda_{33} = \left[\frac{M}{K}\cos\beta L + (\kappa-1)\sin\beta L\right]e^{-2\alpha L}, \\
&\Lambda_{34} = -\left[\frac{M}{K}\cos\beta L + (\kappa+1)\sin\beta L\right], \\
&\Lambda_{41} = \Lambda_{33}, \quad \Lambda_{42} = \Lambda_{34}, \quad \Lambda_{43} = -\Lambda_{31}, \quad \Lambda_{44} = -\Lambda_{32}.
\end{aligned}\right\} \tag{101}$$

Obviously, the resonance occurs only when

$$\det \boldsymbol{\Lambda} = 0. \tag{102}$$

After some manipulation, (102) becomes

$$\left[(\kappa-1)^2+\left(\frac{M}{K}\right)^2\right]^2 \mathrm{e}^{-4\alpha L}+\left[(\kappa+1)^2+\left(\frac{M}{K}\right)^2\right]^2$$
$$+2\left\{4\kappa^2\left(\frac{M}{K}\right)^2-\left[1+\left(\frac{M}{K}\right)^2-\kappa^2\right]^2\right\}\mathrm{e}^{-2\alpha L}\cos 2\,\beta L$$
$$-8\kappa\frac{M}{K}\left[1+\left(\frac{M}{K}\right)^2-\kappa^2\right]\mathrm{e}^{-2\alpha L}\sin 2\beta L = 0,$$

from which the resonance condition can be written as

$$\sin(2\beta L+\Delta) = \frac{\left[(\kappa-1)^2+\left(\frac{M}{K}\right)^2\right]^2 \mathrm{e}^{-2\alpha L}+\left[(\kappa+1)^2+\left(\frac{M}{K}\right)^2\right]^2 \mathrm{e}^{2\alpha L}}{2\left\{4\kappa^2\left(\frac{M}{K}\right)^2+\left[1+\left(\frac{M}{K}\right)^2-\kappa^2\right]^2\right\}}, \tag{103}$$

where

$$\tan\Delta = \frac{4\kappa^2\left(\frac{M}{K}\right)^2-\left[1+\left(\frac{M}{K}\right)^2-\kappa^2\right]}{4\kappa\frac{M}{K}\left[1+\left(\frac{M}{K}\right)^2-\kappa^2\right]}. \tag{104}$$

Since $|\sin(2\beta L+\Delta)| \leqslant 1$, the right-hand side of (103) must be less than or equal to unity for possible occurrence of the resonance. It is easy to see that the right-hand side of (103) is always greater than unity, which indicates that the resonance can never occur and that the gas–particle screen is always stable with respect to the acoustic perturbation.

10. Sample calculation

For the numerical calculation, a gas–particle mixture composed of air and solid alumina (Al_2O_2) particles is considered. The alumina particles are produced, for example, in the combustion process in rocket motors with solid propellant. Here we return to the original primed notation for the variables defined in (31)–(33), (51) and (54). The physical constants and the reference conditions are listed in table 1.

Now, one remaining task is to specify the particle distribution function $\phi_0(r_\mathrm{p})$. Practically, however, it is very difficult to specify it, and, at least up to now, there have been few data for the size distribution of solid alumina particles. In the present analysis, therefore, we consider the function $\phi_0(r_\mathrm{p})$ in the form

$$\phi_0(r_\mathrm{p}) = D r_\mathrm{p}^d \exp(-E r_\mathrm{p}^e), \tag{105}$$

where E, d and e are properly specified constants, and the constant D is to be determined for the specified values of E, d and e from the normalizing condition (3). Here, we consider two sets of values:

case (a)

$$D = 2.635\times10^{-24}\ \mathrm{m}^4,\quad d=-5.0,\quad E = 1.6\times10^{-12}\ \mathrm{m}^2,\quad e=-2.0;$$

Physical constants

Air	Al_2O_3
$\gamma = 1.4$	$\rho_p = 4.0 \times 10^3$ kg/m^3
$C_{pg} = 1005.0$ J/kg K	$C_{pp} = 1686.0$ J/kg K
$\mu = 1.79 \times 10^{-5}$ kg/m s (for $T_0 = 288.0$ K)	
$P_r = 0.75$	

Reference conditions

$T_0 = 288.0$ K
$\rho_0 = 1.23$ kg/m^3
$a_{f0} = 340.0$ m/s

TABLE 1

case (b)

$$D = 0.2857 \times 10^6\ \mathrm{m^{-1}}, \quad d = 0, \quad E = 0, \quad e = 0;$$

where it is assumed that the minimum and the maximum particle radii $r_{p\,min}$ and $r_{p\,max}$ of the particles contained in the mixture are specified as

$$r_{p\,min} = 0.5\ \mu\mathrm{m}, \quad r_{p\,max} = 4.0\ \mu\mathrm{m}.$$

According to the present specification of $r_{p\,min}$ and $r_{p\,max}$ given above, we can put

$$\int (\ )\,dr_p = \int_{r_{p\,min}}^{r_{p\,max}} (\ )\,dr_p, \quad \int (\ )\,dr'_p = \int_{r'_{p\,min}}^{r'_{p\,max}} (\ )\,dr'_p$$

for all the integrations over particle radii in the system, where $r'_{p\,min}$ and $r'_{p\,max}$ are $r_{p\,min}/l_p$ and $r_{p\,max}/l_p$ respectively. The average particle radii l_p defined by (9) are

$$l_p = \begin{cases} 1.353\ \mu\mathrm{m} & \text{for case } (a) \\ 2.615\ \mu\mathrm{m} & \text{for case } (b). \end{cases}$$

Case (*a*) was determined from the experimental data of Kliegel (1963) by curve fitting. Case (*b*) is only a fictitious uniform distribution, and is considered here in order to investigate the effect of the form of $\phi_0(r_p)$ on the wave phenomena in the mixture by comparing the results for cases (*a*) and (*b*). Figure 2 shows $\phi_0(r_p)$ for these two cases.

The numerical results of the coefficients of transmission, absorption and reflection are plotted respectively in figures 3, 4 and 5 against the width L' (or L) of the gas–particle screen. These results are for $\nu = 1$ and $f = 1000\ \mathrm{s^{-1}}$, where f is the frequency of the incident wave and is related to the characteristic time τ by

$$\tau = \frac{1}{\omega} = \frac{1}{2\pi f}. \tag{106}$$

Appreciable differences between the results for cases (*a*) and (*b*) are well seen in these figures. This indicates the importance of detailed knowledge of the structure of $\phi_0(r_p)$ in the analysis of acoustic response of the gas–particle screen. These figures also suggest that the gas–particle screen is relatively far more effective for the absorption than for the reflection of acoustic disturbances. The effect of the geometry of gas–particle screen appears most strongly on the coefficient of reflection α_r. The difference between the results for cases (*a*) and (*b*) is very prominent for α_r. This

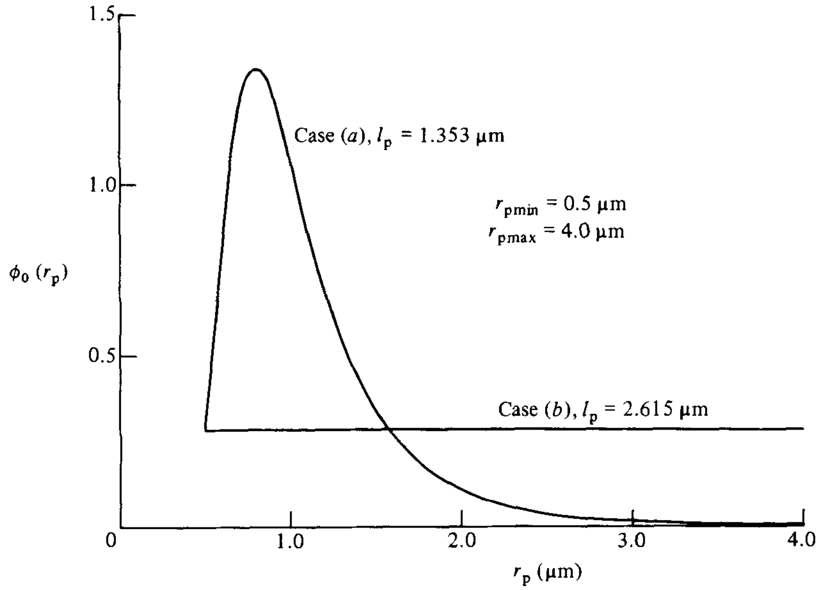

FIGURE 2. Particle distribution function.

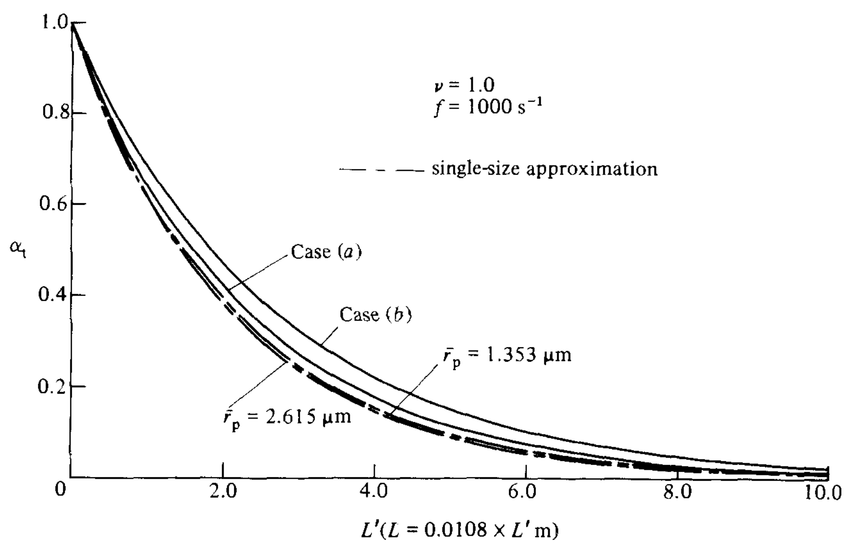

FIGURE 3. Coefficient of transmission.

geometric effect of the gas–particle screen is naturally reflected in the coefficients of transmission and absorption, but its contribution to these coefficients is very small.

It would be a very interesting and important problem to find an equivalent or best-fit single-size-particle dusty-gas mixture that gives good approximations to the coefficients of transmission, absorption and reflection of the corresponding gas–particle mixture with the actual size distribution $\phi_0(r_p)$. Since the present analytical result remains valid for the mixture with a single size of particles by putting

$$\phi_0(r_p) = \delta(r_p - \bar{r}_p), \tag{107}$$

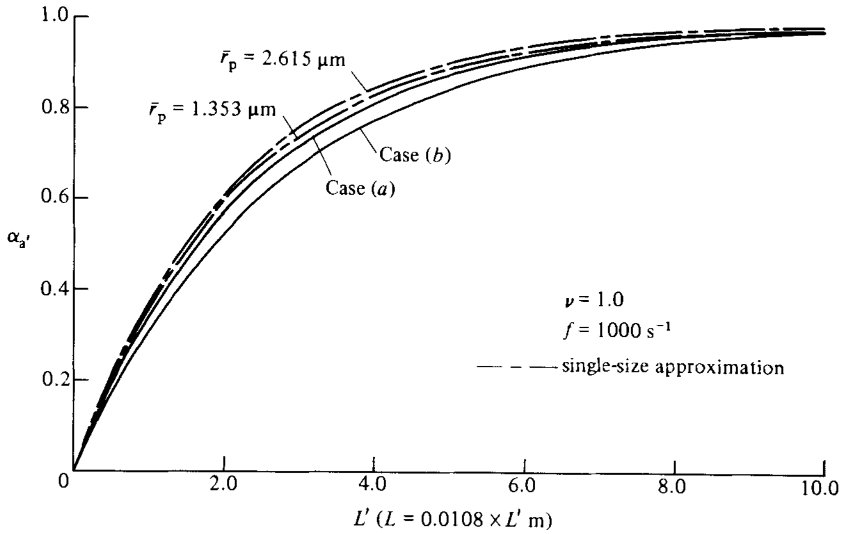

FIGURE 4. Coefficient of absorption.

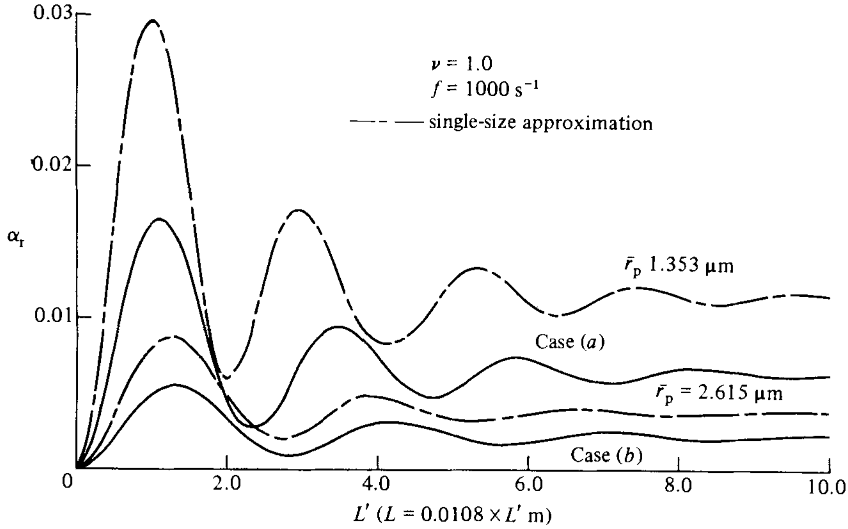

FIGURE 5. Coefficient of reflection.

where δ is the Dirac's delta-function, the problem is equivalent to finding a particle size $\bar{r}_p$ that satisfies the four equations

$$\frac{\bar{A}_p'^k}{\bar{A}_p'^2+1} = \hat{A}_{pk}, \quad \frac{\bar{B}_p'^k}{\bar{B}_p'^2+1} = \hat{B}_{pk} \quad (k = 0, 1),$$

where $\hat{A}_{pk}$ and $\hat{B}_{pk}$ are defined by (69), and $\bar{A}_p'$ and $\bar{B}_p'$ are A_p' and B_p' evaluated for $r_p = \bar{r}_p$. Obviously it is impossible mathematically to find such a $\bar{r}_p$. Hence, therefore, a plausible single-size approximation is proposed by taking

$$\bar{r}_p = l_p,$$

where l_p is defined by (9) and its value has been obtained previously for each case of size distribution.

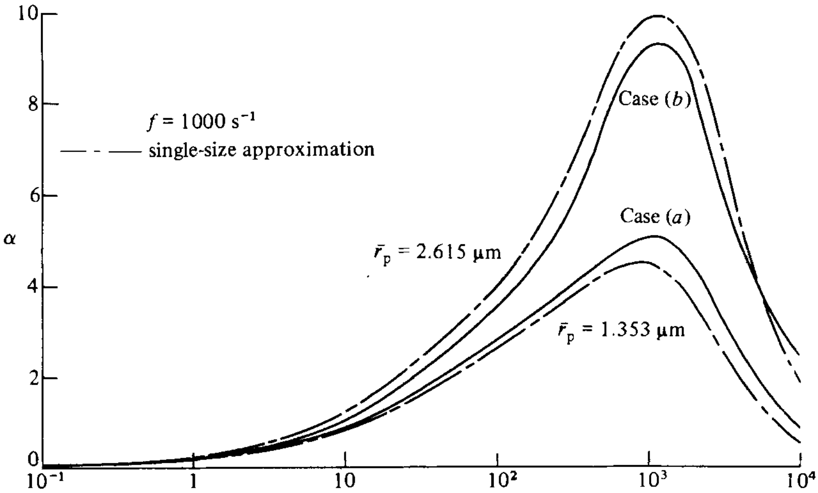

FIGURE 6. Parameter α.

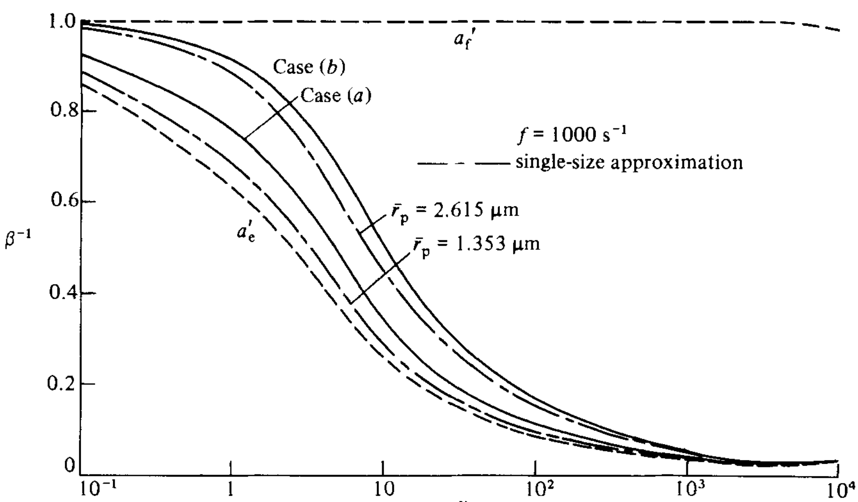

FIGURE 7. Speed of sound in the mixture.

The results for $\phi_0(r_p) = \delta(r_p - l_p)$ are shown in figures 3–5. It is reasonable to expect that the accuracy of this single-size approximation depends on the form of the corresponding actual distribution function $\phi_0(r_p)$. In the present calculation, the single-size results for case (a) give relatively better approximations than those for case (b). The reason can be seen clearly from figure 2. In case (a), the majority of particles are distributed in the radius range near the corresponding average particle radius l_p. For both cases, however, the present single-size approximation gives very poor results for the coefficient of reflection α_r, as shown in figure 5.

The accuracy of the single-size approximation proposed here can also be investigated by comparing the parameters α and β, which are defined by (83). The former concerns the absorption of the disturbance, and the latter represents the inverse velocity of sound in the mixture. These are plotted against the loading ratio ν in figures 6 and 7. The present single-size approximation seems to give a relatively poorer result for

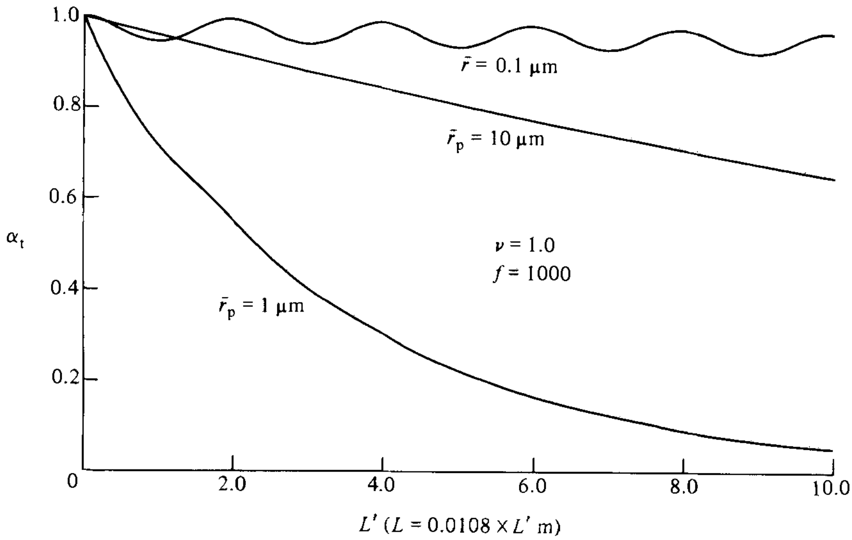

FIGURE 8. Coefficient of transmission for a single size of particles.

β (or $1/\beta$) than for α. These figures also indicate, as has been suggested previously, that the form of the distribution function $\phi_0(r_p)$ has a very strong effect on the parameters α and β. In figures 6 and 7, a_f' and a_e' are the sound velocities non-dimensionalized by a_{f0} in the mixture in the frozen and equilibrium limits respectively. The parameters α in these two limiting cases are both zero, which means that, in these limiting cases, the absorption of the disturbance by the gas–particle mixture does not occur. This result coincides with the situation in the vibrationally or chemically relaxing gases.

The effect of the particle volume ratio ϵ_0 appears, especially in figure 6, for $\nu > 100$. The decreasing tendency in α for $\nu > 1000$ clearly demonstrates the effect of ϵ_0.

It is very important to investigate the particle-size effect on the acoustic response of the gas–particle screen, because the sensitivity of the coefficients α_t, α_a, and α_r to the particle distribution function $\phi_0(r_p)$ seems to come from the significant differences between the acoustic responses of different sizes of particles contained in the mixture. Then the coefficients α_t, α_a and α_r for gas–particle mixtures with only a single size of particles, $\phi_0(r_p) = \delta(r_p - \bar{r}_p)$, have been calculated. The results for $\bar{r}_p = 0.1$, 1.0 and 10.0 μm are shown in figures 8, 9 and 10. It will be very easy to see that there is very strong dependence of the acoustic response of the gas–particle screen on the particle sizes. Especially, it is very important to realize that for a specified frequency of the incident wave, there is some range of particle radii for which the absorption coefficient becomes maximum. For the present conditions, $\nu = 1$ and $f = 1000\ \mathrm{s}^{-1}$, the most effective particle sizes are $r_p = O(1\ \mu\mathrm{m})$. For the acoustic reflection, the smaller sizes of particles are, in general, more effective than the larger sizes of particles. This situation is seen clearly in figures 5 and 10.

Finally, it will be worthwhile to point out the similarity of the present solution for the gas–particle mixture with only a single size of particles. Since A_p and B_p are proportional to r_p^{-2}, and τ is proportional to f^{-1}, we have

$$A_p' \propto \frac{1}{fr_p^2}, \quad B_p' \propto \frac{1}{fr_p^2}.$$

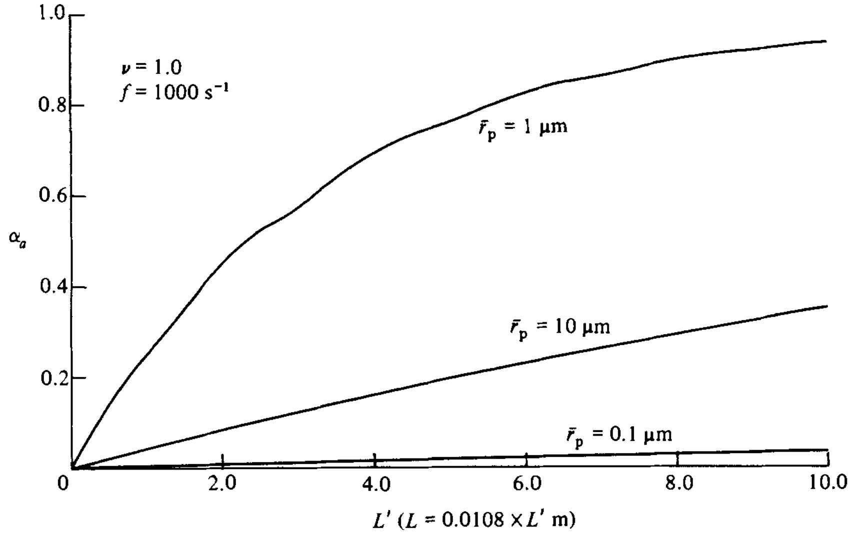

FIGURE 9. Coefficient of absorption for a single size of particles.

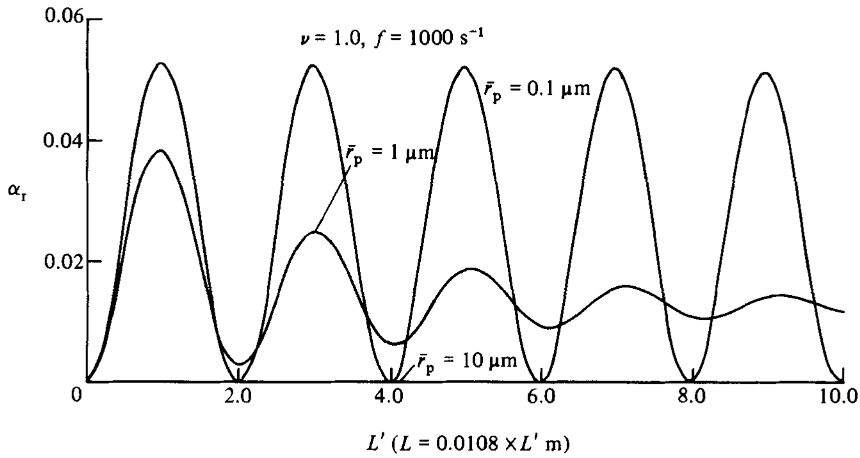

FIGURE 10. Coefficient of reflection for a single size of particles.

Then the present solution is similar for the cases

$$f r_p^2 = \text{const.}$$

This is, however, not the case for the mixture with a continuous size distribution.

11. Conclusions

An acoustic solution for the gas–particle mixture has been obtained completely analytically by taking into account the continuous distribution of particle radii and the finite volume fraction of the particles. It has been proved that the effect of the continuous distribution of particle radii on the wave phenomena is introduced only through four integral quantities of A_p and B_p multiplied by $\phi_0(r_p)\, r_p^3$ over all particle radii: $\hat{A}_{p0}$, $\hat{A}_{p1}$, $\hat{B}_{p0}$ and $\hat{B}_{p1}$. This solution has been applied to the problem of acoustic reflection, transmission and absorption by the gas–particle screen.

The numerical results have shown that the acoustic response of the gas–particle screen depends strongly on the particle sizes. The need for detailed knowledge of the size distribution $\phi_0(r_p)$ is then inevitable for the precise prediction of the coefficients of reflection, transmission and absorption by the gas–particle screen.

Appendix

From (34), we can get

$$\frac{\rho_0}{\rho_p} = \frac{1}{1-\epsilon_0}\frac{\epsilon_0}{\nu}, \tag{A 1}$$

which yields, in conjunction with assumption (iv) of §2,

$$0 < \frac{\epsilon_0}{\nu} < 1-\epsilon_0, \tag{A 2}$$

since the condition

$$0 < \epsilon_0 < 1 \tag{A 3}$$

is always satisfied. Using these relations, it follows from the expressions for $\hat{C}_{p1}$ and $\hat{C}_{p2}$, which can be obtained by substituting (71) into (68), that

$$\hat{C}_{p1} > 0, \tag{A 4}$$

$$\hat{C}_{p2} = -\frac{\hat{A}_{p1}\left[(1-\epsilon_0)-\dfrac{\epsilon_0}{\nu}\right]}{(1-\epsilon_0\hat{A}_{p0})^2+\epsilon_0^2\hat{A}_{p1}^2} < 0, \tag{A 5}$$

where use has been made of (70).

Obviously, the sign of Q is equal to that of $a_1a_4-a_2a_3$, which can be obtained with (76) as

$$\frac{a_1a_4-a_2a_3}{1-\epsilon_0} = (\gamma-1)\,\nu\theta\hat{B}_{p1}\left(1+\frac{\epsilon_0}{1-\epsilon_0}\hat{C}_{p1}\right)(1+\nu\hat{C}_{p1})+(\gamma-1)\frac{\epsilon_0}{1-\epsilon_0}\nu^2\theta\hat{B}_{p1}\hat{C}_{p2}^2$$
$$-\nu[\gamma\nu^2\theta^2\hat{B}_{p1}^2+(1+\nu\theta\hat{B}_{p2})(1+\gamma\nu\theta\hat{B}_{p2})]\left(1-\frac{\epsilon_0}{1-\epsilon_0}\frac{1}{\nu}\right)\hat{C}_{p2}. \tag{A 6}$$

Equations (A 2)–(A 5), together with the fact that γ is always greater than unity, imply that

$$a_1a_4-a_2a_3 > 0, \tag{A 7}$$

or, as was to be shown,

$$Q > 0. \tag{A 8}$$

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